Twisted partially pure spinors

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Abstract

Motivated by the relationship between orthogonal complex structures and spure spinors, we define twisted partially pure spinors in order to characterize spinorially subspaces of Euclidean space endowed with a complex structure.

1 Introduction

In this paper, we characterize subspaces of Euclidean space $\mathbb{R}^n$ endowed with an orthogonal complex structure by means of twisted spinors, which is a generalization of the relation between classical pure spinors and orthogonal complex structures on Euclidean space $\mathbb{R}^{2m}$. Recall that a classical pure spinor $\phi \in \Delta_{2m}$ is a spinor such that the (isotropic) subspace of complexified vectors $X - iY \in \mathbb{R}^{2m} \otimes \mathbb{C}$, $X, Y \in \mathbb{R}^{2m}$, which annihilate ϕ under Clifford multiplication

$$(X - iY) \cdot \phi = 0$$

is of maximal dimension, where $m \in \mathbb{N}$ and Δ_{2m} is the standard complex representation of the Spin group $Spin(2m)$ (cf. [5]). This means that for every $X \in \mathbb{R}^{2m}$ there exists a $Y \in \mathbb{R}^{2m}$ satisfying

$$X \cdot \phi = iY \cdot \phi.$$

By setting $Y = J(X)$, one can see that a pure spinor determines a complex structure on $\mathbb{R}^{2m}$. Geometrically, the two subspaces $TM \cdot \phi$ and $iTM \cdot \phi$ of Δ_{2m} coincide, which means $TM \cdot \phi$ is a complex subspace of Δ_{2m}, and the effect of multiplication by the number $i = \sqrt{-1}$ is transferred to the tangent space TM in the form of J.

The authors of [1, 6] investigated (the classification of) non-pure classical spinors by means of their isotropic subspaces. In [6], the authors noted that there may be many spinors (in different orbits under the action of the Spin group) admitting isotropic subspaces of the same dimension, and that there is a gap in the possible dimensions of such isotropic subspaces. In our Euclidean/Riemannian context, such isotropic subspaces correspond to subspaces of Euclidean space endowed with orthogonal complex structures. In this paper, we define twisted partially pure

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spinors (cf. Definition 3.1) in order to establish a one-to-one correspondence between subspaces of Euclidean space (of a fixed codimension) endowed with orthogonal complex structures (and oriented orthogonal complements), and orbits of such spinors under a particular subgroup of the twisted spin group (cf. Theorem 3.1). By using spinorial twists we avoid having different orbits under the full twisted spin group and also the aforementioned gap in the dimensions.

The need to establish such a correspondence arises from our interest in developing a spinorial setup to study the geometry of manifolds admitting (almost) CR structures (of arbitrary codimension) and elliptic structures. Since such manifolds are not necessarily Spin nor Spin^c, we are led to consider spinorially twisted spin groups, representations, structures, etc. Geometric and topological considerations regarding such manifolds will be presented in [4].

The paper is organized as follows. In Section 2 we recall basic material on Clifford algebras, spin groups and representations; we define the twisted spin groups and representations that will be used, and the space of anti-symmetric 2-forms and endomorphims associated to twisted spinors; we also present some results on subgroups and branching of representations. In Section 3, we define partially pure spinors, deduce their basic properties and prove the main theorem, Theorem 3.1, which establishes the aforementioned one-to-one correspondence.

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2 Preliminaries

ec:preliminaries⟩ In this section, we briefly recall basic facts about Clifford algebras, the Spin group and the standard Spin representation [3]. We also define the twisted spin groups and representations, and the antisymmetric 2-forms and endomorphisms associated to a twisted spinor, and describe various inclusions of groups into (twisted) spin groups.

2.1 Clifford algebras

Let Cl_n denote the Clifford algebra generated by the orthonormal vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_j e_k + e_k e_j \quad = \quad -2\langle e_j, e_k\rangle,$$

where $\langle , \rangle$ denotes the standard inner product in $\mathbb{R}^n$. Let

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$$

denote the complexification of Cl_n. The Clifford algebras are isomorphic to matrix algebras. In particular,

$$\mathbb{C}l_n \cong \begin{cases} \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k, \\ \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k+1, \end{cases}$$

where

$$\Delta_n := \mathbb{C}^{2^k} = \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{k \text{ times}}$$

is the tensor product of $k=[\frac{n}{2}]$ copies of $\mathbb{C}^2$. The map

$$\kappa : \mathbb{C}l_n \longrightarrow \mathrm{End}(\mathbb{C}^{2^k})$$

is defined to be either the above mentioned isomorphism if n is even, or the isomorphism followed by the projection onto the first summand if n is odd. In order to make κ explicit, consider the following matrices

$$Id=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In terms of the generators $e_1,\ldots,e_n$, κ can be described explicitly as follows,

$$\begin{array}{rcl} e_1 & \mapsto & Id\otimes Id\otimes \ldots \otimes Id\otimes Id\otimes g_1, \\ e_2 & \mapsto & Id\otimes Id\otimes \ldots \otimes Id\otimes Id\otimes g_2, \\ e_3 & \mapsto & Id\otimes Id\otimes \ldots \otimes Id\otimes g_1\otimes T, \\ e_4 & \mapsto & Id\otimes Id\otimes \ldots \otimes Id\otimes g_2\otimes T, \\ \vdots & & \ldots \\ e_{2k-1} & \mapsto & g_1\otimes T\otimes \ldots \otimes T\otimes T\otimes T, \\ e_{2k} & \mapsto & g_2\otimes T\otimes \ldots \otimes T\otimes T\otimes T, \end{array}$$

and, if $n=2k+1$,

$$e_{2k+1}\mapsto i\,T\otimes T\otimes \ldots \otimes T\otimes T\otimes T.$$

The vectors

$$u_{+1}=\frac{1}{\sqrt{2}}(1,-i) \qquad \text{and} \qquad u_{-1}=\frac{1}{\sqrt{2}}(1,i),$$

form a unitary basis of $\mathbb{C}^2$ with respect to the standard Hermitian product. Thus,

$$\{u_{\varepsilon_1,\ldots,\varepsilon_k}=u_{\varepsilon_1}\otimes\ldots\otimes u_{\varepsilon_k}\,|\,\varepsilon_j=\pm 1, j=1,\ldots,k\},$$

is a unitary basis of $\Delta_n=\mathbb{C}^{2^k}$ with respect to the naturally induced Hermitian product. We will denote inner and Hermitian products by the same symbol $\langle\cdot,\cdot\rangle$ trusting that the context will make clear which product is being used.

Clifford multiplication is defined by

$$\begin{array}{rcl} \mu_n : \mathbb{R}^n\otimes\Delta_n & \longrightarrow & \Delta_n \\ x\otimes\psi & \mapsto & \mu_n(x\otimes\psi)=x\cdot\psi:=\kappa(x)(\psi). \end{array}$$

A quaternionic structure α on $\mathbb{C}^2$ is given by

$$\alpha\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}=\begin{pmatrix} -\overline{z}_2 \\ \overline{z}_1 \end{pmatrix},$$

and a real structure β on $\mathbb{C}^2$ is given by

$$\beta\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}=\begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \end{pmatrix}.$$

Real and quaternionic structures γ_n on $\Delta_n = (\mathbb{C}^2)^{\otimes[n/2]}$ are built as follows

$$\begin{array}{rcllr}
\gamma_n &=& (\alpha\otimes\beta)^{\otimes 2k} & \text{if } n = 8k, 8k+1 & \text{(real)},\\
\gamma_n &=& \alpha\otimes(\beta\otimes\alpha)^{\otimes 2k} & \text{if } n = 8k+2, 8k+3 & \text{(quaternionic)},\\
\gamma_n &=& (\alpha\otimes\beta)^{\otimes 2k+1} & \text{if } n = 8k+4, 8k+5 & \text{(quaternionic)},\\
\gamma_n &=& \alpha\otimes(\beta\otimes\alpha)^{\otimes 2k+1} & \text{if } n = 8k+6, 8k+7 & \text{(real)}.
\end{array}$$

2.2 The Spin group and representation

The Spin group $Spin(n) \subset Cl_n$ is the subset

$$Spin(n) = \{x_1x_2\cdots x_{2l-1}x_{2l}\,|\,x_j \in \mathbb{R}^n,\ |x_j| = 1,\ l \in \mathbb{N}\},$$

endowed with the product of the Clifford algebra. It is a Lie group and its Lie algebra is

$$\mathfrak{spin}(n) = \mathrm{span}\{e_ie_j\,|\,1 \le i < j \le n\}.$$

Recall that the Spin group $Spin(n)$ is the universal double cover of $SO(n)$, $n \ge 3$. For $n = 2$ we consider $Spin(2)$ to be the connected double cover of $SO(2)$. The covering map will be denoted by

$$\lambda_n : Spin(n) \to SO(n).$$

Its differential is given by $\lambda_{n_*}(e_ie_j) = 2E_{ij}$, where $E_{ij} = e_i^* \otimes e_j - e_j^* \otimes e_i$ is the standard basis of the skew-symmetric matrices, and e^* denotes the metric dual of the vector e. Furthermore, we will abuse the notation and also denote by λ_n the induced representation on $\bigwedge^*\mathbb{R}^n$.

The restriction of κ to $Spin(n)$ defines the Lie group representation

$$Spin(n) \longrightarrow GL(\Delta_n),$$

which is, in fact, special unitary. We have the corresponding Lie algebra representation

$$\mathfrak{spin}(n) \longrightarrow \mathfrak{gl}(\Delta_n).$$

Remark. For the sake of notation we will set

$$SO(0) = \{1\}, \qquad SO(1) = \{1\},$$

$$Spin(0) = \{\pm 1\}, \qquad Spin(1) = \{\pm 1\},$$

and

$$\Delta_0 = \Delta_1 = \mathbb{C}$$

a trivial 1-dimensional representation.

Clifford multiplication μ_n has the following properties:

- It is skew-symmetric with respect to the Hermitian product

$$\langle x\cdot\psi_1, \psi_2\rangle = -\langle \psi_1, x\cdot\psi_2\rangle\,. \qquad (1)$$ clifford-skew-s

- μ_n is an equivariant map of $Spin(n)$ representations.

- μ_n can be extended to an equivariant map

$$\begin{array}{rcl}\mu_n : \bigwedge^*(\mathbb{R}^n)\otimes\Delta_n & \longrightarrow & \Delta_n \\ \omega\otimes\psi & \mapsto & \omega\cdot\psi,\end{array}$$

of $Spin(n)$ representations.

At this point we will make the following convention. Consider the involution

$$\begin{array}{rcl}F_{2m} : \Delta_{2m} & \longrightarrow & \Delta_{2m} \\ \phi & \mapsto & (-i)^m e_1e_2\cdots e_{2m}\cdot\phi,\end{array}$$

and let

$$\Delta_{2m}^{\pm} = \{\phi \,|\, F_{2m}(\phi) = \pm\phi\}.$$

This definition of positive and negative Weyl spinors differs from the one in [3] by a factor $(-1)^m$. Nevertheless, we have chosen this convention so that the spinor $u_{1,\ldots,1}$ is always positive and corresponds to the standard (positive) complex structure on $\mathbb{R}^{2m}$.

2.3 Spinorially twisted Spin groups

Consider the following groups:

- By using the unit complex numbers $U(1)$, the Spin group can be twisted [3]

$$Spin^c(n) = (Spin(n)\times U(1))/\{\pm(1,1)\} = Spin(n)\times_{\mathbb{Z}_2} U(1),$$

  with Lie algebra

$$\mathfrak{spin}^c(n) = \mathfrak{spin}(n)\oplus i\mathbb{R}.$$

- In [2] we have considered the twisted Spin group $Spin^r(n)$, $r\in\mathbb{N}$, defined as follows

$$Spin^r(n) = (Spin(n)\times Spin(r))/\{\pm(1,1)\} = Spin(n)\times_{\mathbb{Z}_2} Spin(r).$$

  The Lie algebra of $Spin^r(n)$ is

$$\mathfrak{spin}^r(n) = \mathfrak{spin}(n)\oplus\mathfrak{spin}(r).$$

- Here, we will also consider the following group

$$\begin{array}{rcl}Spin^{c,r}(n) & = & (Spin(n)\times Spin^c(r))/\{\pm(1,1)\} \\ & = & Spin(n)\times_{\mathbb{Z}_2} Spin^c(r),\end{array}$$

  where $r\in\mathbb{N}$, whose Lie algebra is

$$\mathfrak{spin}^c(n) = \mathfrak{spin}(n)\oplus\mathfrak{spin}(r)\oplus i\mathbb{R}.$$

  It fits into the exact sequence

$$1\longrightarrow\mathbb{Z}_2\longrightarrow Spin^{c,r}(n)\xrightarrow{\lambda_n\times\lambda_r\times\lambda_2} SO(n)\times SO(r)\times U(1)\longrightarrow 1,$$

  where

$$(\lambda_n\times\lambda_r\times\lambda_2)([g,[h,z]]) \;=\; (\lambda_n(g),\lambda_r(h),z^2).$$

Remark. For $r=0,1$, $Spin^{c,r}(n) = Spin^c(n)$.

2.4 Twisted spin representations

Consider the following twisted representations:

- The Spin representation Δ_n extends to a representation of $Spin^c(n)$ by letting

$$\begin{array}{rcl} Spin^c(n) & \longrightarrow & GL(\Delta_n) \\ [g,z] & \mapsto & z\kappa_n(g) =: zg. \end{array}$$

- The twisted $Spin^{c,r}(n)$ representation

$$\begin{array}{rcl} Spin^{c,r}(n) & \longrightarrow & GL(\Delta_r \otimes \Delta_n) \\ [g,[h,z]] & \mapsto & z\,\kappa_r(h) \otimes \kappa_n(g) =: zh \otimes g. \end{array}$$

  which is also unitary with respect to the natural Hermitian metric.

- For $r = 0, 1$, the twisted spin representation is simply the $Spin^c(n)$ representation Δ_n.

  We will also need the map

$$\begin{array}{rcl} \mu_r \otimes \mu_n : \left(\bigwedge^* \mathbb{R}^r \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R}^n\right) \otimes_{\mathbb{R}} (\Delta_r \otimes \Delta_n) & \longrightarrow & \Delta_r \otimes \Delta_n \\ (w_1 \otimes w_2) \otimes (\psi \otimes \varphi) & \mapsto & (w_1 \otimes w_2) \cdot (\psi \otimes \varphi) = (w_1 \cdot \psi) \otimes (w_2 \cdot \varphi). \end{array}$$

As in the untwisted case, $\mu_r \otimes \mu_n$ is an equivariant homomorphism of $Spin^{c,r}(n)$ representations.

2.5 Skew-symmetric 2-forms and endomorphisms associated to twisted spinors

We will often write f_{kl} for the Clifford product $f_k f_l$.

Definition 2.1 [2] *Let $r \geq 2$, $\phi \in \Delta_r \otimes \Delta_n$, $X, Y \in \mathbb{R}^n$, $(f_1 \ldots, f_r)$ an orthonormal basis of $\mathbb{R}^r$ and $1 \leq k, l \leq r$.*

- *Define the real 2-forms associated to the spinor ϕ by*

$$\eta^\phi_{kl}(X,Y) = \mathrm{Re}\,\langle X \wedge Y \cdot f_k f_l \cdot \phi, \phi\rangle\,.$$

- *Define the antisymmetric endomorphisms $\hat{\eta}^\phi_{kl} \in \mathrm{End}^-(\mathbb{R}^n)$ by*

$$X \mapsto \hat{\eta}^\phi_{kl}(X) := (X \lrcorner \eta^\phi_{kl})^\sharp,$$

  where $X \in \mathbb{R}^n$, $\lrcorner$ denotes contraction and $^\sharp$ denotes metric dualization from 1-fomrs to vectors.

Lemma 2.1 *Let $r \geq 2$, $\phi \in \Delta_r \otimes \Delta_n$, $X, Y \in \mathbb{R}^n$, $(f_1 \ldots, f_r)$ an orthonormal basis of $\mathbb{R}^r$ and $1 \leq k, l \leq r$. Then*

$$\begin{array}{rcll} \mathrm{Re}\,\langle f_k f_l \cdot \phi, \phi\rangle & = & 0, & \\ \mathrm{Re}\,\langle X \wedge Y \cdot \phi, \phi\rangle & = & 0, & (2)\ \boxed{\texttt{vanishing3}} \\ \mathrm{Im}\,\langle X \wedge Y \cdot f_k f_l \cdot \phi, \phi\rangle & = & 0, & (3)\ \boxed{\texttt{vanishing4}} \\ \mathrm{Re}\,\langle X \cdot \phi, Y \cdot \phi\rangle & = & \langle X, Y\rangle\,|\phi|^2, & (4)\ \boxed{\texttt{real-part}} \end{array}$$

Proof. By using (1) twice

$$\langle f_k f_l \cdot \phi, \phi \rangle \quad = \quad -\overline{\langle f_k f_l \phi, \phi \rangle}.$$

For identity (2), recall that for $X, Y \in \mathbb{R}^n$

$$X \wedge Y = X \cdot Y + \langle X, Y \rangle \,.$$

Thus

$$\langle X \wedge Y \cdot \phi, \phi \rangle \quad = \quad -\overline{\langle X \wedge Y \cdot \phi, \phi \rangle}.$$

Identities (3) and (4) follow similarly. ◻

Remarks.

- For $k \neq l$,
$$\eta^\phi_{kl} = (\delta_{kl} - 1)\eta^\phi_{lk}.$$
- By (3), if $k \neq l$,
$$\eta^\phi_{kl}(X, Y) = \langle X \wedge Y \cdot f_k f_l \cdot \phi, \phi \rangle \,.$$

Lemma 2.2 [2] *Any spinor $\phi \in \Delta_r \otimes \Delta_n$, $r \geq 2$, defines two maps (extended by linearity)*

$$\begin{array}{ccc} \bigwedge^2 \mathbb{R}^r & \longrightarrow & \bigwedge^2 \mathbb{R}^n \\ f_{kl} & \mapsto & \eta^\phi_{kl} \end{array}$$

and

$$\begin{array}{ccc} \bigwedge^2 \mathbb{R}^r & \longrightarrow & \mathrm{End}(\mathbb{R}^n) \\ f_{kl} & \mapsto & \hat{\eta}^\phi_{kl}, \end{array}$$

◻

2.6 Subgroups, isomorphisms and decompositions

In this section we will describe various inclusions of groups into (twisted) spin groups.

⟨lemma-subgroup1⟩ **Lemma 2.3** *There exists a monomorphism $h : Spin(2m) \times_{\mathbb{Z}_2} Spin(r) \longrightarrow Spin(2m + r)$ such that the following diagram commutes*

$$\begin{array}{ccc} Spin(2m) \times_{\mathbb{Z}_2} Spin(r) & \stackrel{h}{\to} & Spin(2m + r) \\ \downarrow & & \downarrow \\ SO(2m) \times SO(r) & \hookrightarrow & SO(2m + r) \end{array}$$

Proof. Consider the decomposition

$$\mathbb{R}^{2m+r} = \mathbb{R}^{2m} \oplus \mathbb{R}^r,$$

and let

$$\begin{aligned} Spin(2m) &= \left\{ \prod_{i=1}^{2s} x_i \in Cl_{2m+r} \,|\, x_i \in \mathbb{R}^{2m}, |x_i| = 1, s \in \mathbb{N} \right\} \quad \subset \quad Spin(2m+r), \\ Spin(r) &= \left\{ \prod_{j=1}^{2t} y_j \in Cl_{2m+r} \,|\, y_j \in \mathbb{R}^{r}, |y_j| = 1, t \in \mathbb{N} \right\} \quad \subset \quad Spin(2m+r). \end{aligned}$$

It is easy to prove that

$$Spin(2m) \cap Spin(r) = \{1, -1\}.$$

Define the homomorphism

$$\begin{array}{rcl} h : Spin(2m) \times_{\mathbb{Z}_2} Spin(r) & \longrightarrow & Spin(2m+r) \\ {[g, g']} & \mapsto & gg'. \end{array}$$

If $[g, g'] \in Spin(2m) \times_{\mathbb{Z}_2} Spin(r)$ is such that

$$gg' = 1 \in Spin(2m+r),$$

then

$$g' = g^{-1} \in Spin(2m) \subset Spin(2m+r),$$

so that

$$g, g' \in Spin(2m) \cap Spin(r) = \{1, -1\}.$$

Hence $[g, g'] = [1, 1]$ and h is injective. □

⟨lemma:subgroup2⟩ **Lemma 2.4** *Let $r \in \mathbb{N}$. There exists an monomorphism $h : U(m) \times SO(r) \hookrightarrow Spin^{c,r}(2m+r)$ such that the following diagram commutes*

$$\begin{array}{ccc} & & Spin^{c,r}(2m+r) \\ & \nearrow & \downarrow \\ U(m) \times SO(r) & \longrightarrow & SO(2m+r) \times SO(r) \times U(1) \end{array}$$

Proof. Suppose we have an orthogonal complex structure on $\mathbb{R}^{2m} \subset \mathbb{R}^{2m+r}$

$$J : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{2m}, \quad J^2 = -\mathrm{Id}_{2m}, \quad \langle \cdot, \cdot \rangle = \langle J \cdot, J \cdot \rangle.$$

The subgroup of $SO(2m+r)$ that respects both the orthogonal decomposition $\mathbb{R}^{2m+r} = \mathbb{R}^{2m} \oplus \mathbb{R}^r$ and J is

$$U(m) \times SO(r) \quad \subset \quad SO(2m) \times SO(r) \quad \subset \quad SO(2m+r).$$

There exists a lift [3]

$$\begin{array}{ccc} & & Spin^c(2m) \\ & \nearrow & \downarrow \\ U(m) & \to & SO(2m)\times U(1) \\ & & \\ A & \mapsto & (A_{\mathbb{R}}, \det_{\mathbb{C}}(A)) \end{array}$$

and we can consider the diagram [2]

$$\begin{array}{ccc} & & Spin(r)\times_{\mathbb{Z}_2} Spin(r) \\ & \nearrow & \downarrow \\ SO(r) & \xrightarrow{\text{diagonal}} & SO(r)\times SO(r) \end{array} .$$

We can put them together as follows

$$\begin{array}{cccc} & & Spin^c(2m)\times_{\mathbb{Z}_2} Spin^r(r) \cong Spin^r(2m)\times_{\mathbb{Z}_2} Spin^c(r) & \hookrightarrow \quad Spin(2m+r)\times_{\mathbb{Z}_2} Spin^c(r) \\ & \nearrow & \downarrow & \\ U(m)\times SO(r) & \hookrightarrow & SO(2m)\times U(1)\times SO(r)\times SO(r) & \end{array}$$

where the last inclusion is due to Lemma 2.3. It is easy to prove that the lift monomorphism $U(m)\times SO(r) \longrightarrow Spin^c(2m)\times_{\mathbb{Z}_2} Spin^r(r)$ exists and there is a natural isomorphism

$$Spin^c(2m)\times_{\mathbb{Z}_2} Spin^r(r) \cong Spin^r(2m)\times_{\mathbb{Z}_2} Spin^c(r).$$

$\square$

⟨factorization⟩ **Lemma 2.5** *Let $r\in\mathbb{N}$. The standard representation Δ_{2m+r} of $Spin(2m+r)$ decomposes as follows*

$$\Delta_{2m+r} = \Delta_r\otimes\Delta_{2m}^+ \ \oplus\ \Delta_r\otimes\Delta_{2m}^-,$$

with respect to the subgroup $Spin(2m)\times_{\mathbb{Z}_2} Spin(r)\subset Spin(2m+r)$.

Proof. Consider the restriction of the standard representation of $Spin(2m+r)$ to

$$Spin(2m)\times_{\mathbb{Z}_2} Spin(r)\subset Spin(2m+r)\longrightarrow Gl(\Delta_{2m+r}).$$

By using the explicit description of a unitary basis of Δ_{2m+r}, we see that the elements of $Spin(2m)$ act on the last m factors of

$$\Delta_{2m+r} = \underbrace{\mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2}_{[r/2]\text{ times}}\otimes\underbrace{\mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2}_{m\text{ times}},$$

as they do on $\Delta_{2m}=\Delta_{2m}^+\oplus\Delta_{2m}^-$. The elements of $Spin(r)$ act as usual on the first $[r/2]$ factors of Δ_r, act trivially on Δ_{2m}^+, and act by multiplication by (-1) on the factor Δ_{2m}^-. $\square$

3 Twisted partially pure spinors

lly-pure-spinors⟩ In order to simplify the statements, we will consider the twisted spin representation

$$\Sigma_r \otimes \Delta_n \subseteq \Delta_r \otimes \Delta_n.$$

where

$$\Sigma_r = \begin{cases} \Delta_r & \text{if } r \text{ is odd,} \\ \Delta_r^+ & \text{if } r \text{ is even,} \end{cases}$$

$n, r \in \mathbb{N}$.

ally-pure-spinor⟩ **Definition 3.1** *Let $(f_1, \ldots, f_r)$ be an orthonormal frame of $\mathbb{R}^r$. A unit-length spinor $\phi \in \Sigma_r \otimes \Delta_n$, $r < n$, is called a* twisted partially pure spinor *if*

- *there exists a $(n-r)$-dimensional subspace $V^\phi \subset \mathbb{R}^n$ such that for every $X \in V^\phi$, there exists a $Y \in V^\phi$ such that*
$$X \cdot \phi = i\, Y \cdot \phi.$$
- *it satisfies the equations*
$$\begin{aligned} (\eta^\phi_{kl} + f_k f_l) \cdot \phi &= 0, \\ \langle f_k f_l \cdot \phi, \phi \rangle &= 0, \end{aligned}$$
for all $1 \leq k < l \leq r$.
- *If $r = 4$, it also satisfies the condition*
$$\langle f_1 f_2 f_3 f_4 \cdot \phi, \phi \rangle = 0.$$

Remarks.

1. The requirement $|\phi| = 1$ is made in order to avoid renormalizations later on.
2. The extra condition for the case $r = 4$ is fulfilled for all other ranks.
3. From now on we will drop the adjective twisted since it will be clear from the context.

3.1 Example of partially pure spinor

⟨lemma:existence⟩ **Lemma 3.1** *Given $r, m \in \mathbb{N}$, there exists a partially pure spinor in $\Sigma_r \otimes \Delta_{2m+r}$.*

Proof. Let $(e_1, \ldots, e_{2m}, e_{2m+1}, \ldots, e_{2m+r})$ and $(f_1, \ldots, f_r)$ be orthonormal frames of $\mathbb{R}^{2m+r}$ and $\mathbb{R}^r$ respectively. Consider the decomposition of Lemma 2.5

$$\Delta_{2m+r} = \Delta_r \otimes \Delta_{2m}^+ \,\oplus\, \Delta_r \otimes \Delta_{2m}^-,$$

corresponding to the decomposition

$$\mathbb{R}^{2m+r} = \mathrm{span}\{e_1, \ldots, e_{2m}\} \oplus \mathrm{span}\{e_{2m+1}, \ldots, e_{2m+r}\}.$$

Let

$$\varphi_0 = u_{1,\ldots,1} \in \Delta_{2m}^+,$$

and

$$\{v_{\varepsilon_1,\ldots,\varepsilon_{[r/2]}}\,|\,(\varepsilon_1,\ldots,\varepsilon_{[r/2]}) \in \{\pm 1\}^{[r/2]}\}$$

be the unitary basis of the twisting factor $\Delta_r = \Delta(\mathrm{span}(f_1,\ldots,f_r))$ which contains Σ_r. Let us define the standard twisted partially pure spinor $\phi_0 \in \Sigma_r \otimes \Delta_r \otimes \Delta_{2m}^+$ by

$$\phi_0 = \begin{cases} \frac{1}{\sqrt{2^{[r/2]}}}\left(\sum_{I\in\{\pm1\}^{\times[r/2]}} v_I \otimes \gamma_r(u_I)\right) \otimes \varphi_0 & \text{if } r \text{ is odd,} \\ \frac{1}{\sqrt{2^{[r/2]-1}}}\left(\sum_{I\in\left[\{\pm1\}^{\times[r/2]}\right]_+} v_I \otimes \gamma_r(u_I)\right) \otimes \varphi_0 & \text{if } r \text{ is even,} \end{cases}$$

where the elements of $\left[\{\pm1\}^{\times[r/2]}\right]_+$ contain an even number of (-1).

Checking the conditions in the definition of partially pure spinor for ϕ_0 is done by a (long) direct calculation as in [2]. For instance, taking $n = 7$, $r = 3$, we have

$$\phi_0 = \frac{1}{\sqrt{2}}(v_1 \otimes \gamma_3(u_1) \otimes u_1 \otimes u_1 + v_{-1} \otimes \gamma_3(u_{-1}) \otimes u_1 \otimes u_1)$$

where γ_3 is a quaternionic structure. We check that this ϕ_0 is a partially pure spinor. Putting $C = 1/\sqrt{2}$ and remembering that $\gamma_3(u_\epsilon) = -i\epsilon u_{-\epsilon}$, we get

$$\phi_0 = iC(v_{-1} \otimes u_1 \otimes u_1 \otimes u_1 - v_1 \otimes u_{-1} \otimes u_1 \otimes u_1),$$

which has unit length. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^7$, so that

$$\begin{aligned} e_1 \cdot \phi_0 &= iC(v_{-1} \otimes u_1 \otimes u_1 \otimes g_1(u_1) - v_1 \otimes u_{-1} \otimes u_1 \otimes g_1(u_1)) \\ &= ie_2 \cdot \phi_0, \end{aligned}$$

and, similarly,

$$e_3 \cdot \phi_0 = ie_4 \cdot \phi_0.$$

So, ϕ_0 induces the standard complex structure on $V^{\phi_0} = \langle e_1, e_2, e_3, e_4 \rangle$. Let $\{f_i\}$ be the standard basis of $\mathbb{R}^3$. Similar calculations give

$$\eta_{kl}^{\phi_0} = e_{4+k} \wedge e_{4+l},$$

$$(\eta_{kl}^{\phi_0} + f_{kl}) \cdot \phi_0 = 0,$$

and

$$\langle f_{kl} \cdot \phi_0, \phi_0 \rangle = 0.$$

$\square$

3.2 Properties of partially pure spinors

basic properties⟩ **Lemma 3.2** *The definition of partially pure spinor does not depend on the choice of orthonormal basis of $\mathbb{R}^r$.*

Proof. If $r = 0, 1$, a partially pure spinor is a classical pure spinor for n even or the straightforward generalization of pure spinor for n odd [5, p. 336]. Suppose $(f_1', \ldots, f_r')$ is another orthonormal frame of $\mathbb{R}^r$, then

$$f_i' = \alpha_{i1} f_1 + \cdots + \alpha_{ir} f_r,$$

so that the matrix $A = (\alpha_{ij}) \in SO(r)$. Let us denote

$$\eta_{kl}'^{\phi}(X,Y) := \mathrm{Re}\left\langle X \wedge Y \cdot f_k' f_l' \cdot \phi, \phi \right\rangle$$

Thus,

$$\begin{aligned}
\eta_{kl}'^{\phi} \cdot \phi &= \sum_{1\leq a<b\leq n} \eta_{kl}'^{\phi}(e_a,e_b) e_a e_b \cdot \phi \\
&= \sum_{1\leq a<b\leq n} \mathrm{Re}\left\langle e_a e_b \cdot \left(\sum_{s=1}^{r} \alpha_{ks} f_s\right)\left(\sum_{t=1}^{r} \alpha_{lt} f_t\right) \cdot \phi, \phi \right\rangle e_a e_b \cdot \phi \\
&= \sum_{1\leq a<b\leq n} \sum_{s=1}^{r}\sum_{t=1}^{r} \alpha_{ks}\alpha_{lt} \mathrm{Re}\left\langle e_a e_b \cdot f_s f_t \cdot \phi, \phi \right\rangle e_a e_b \cdot \phi \\
&= \sum_{1\leq a<b\leq n} \sum_{s=1}^{r}\sum_{t=1}^{r} \alpha_{ks}\alpha_{lt} \eta_{st}^{\phi}(e_a,e_b) e_a e_b \cdot \phi \\
&= \sum_{s=1}^{r}\sum_{t=1}^{r} \alpha_{ks}\alpha_{lt} \eta_{st}^{\phi} \cdot \phi \\
&= -\sum_{s=1}^{r}\sum_{t=1}^{r} \alpha_{ks}\alpha_{lt} f_s f_t \cdot \phi \\
&= -\left(\sum_{s=1}^{r} \alpha_{ks} f_s\right)\left(\sum_{t=1}^{r} \alpha_{lt} f_t\right) \cdot \phi \\
&= -f_k' f_l' \cdot \phi.
\end{aligned}$$

For the third part of the definition, note that

$$\begin{aligned}
\left\langle f_k' f_l' \cdot \phi, \phi \right\rangle &= \left\langle \left(\sum_{s=1}^{r} \alpha_{ks} f_s\right)\left(\sum_{t=1}^{r} \alpha_{lt} f_t\right) \cdot \phi, \phi \right\rangle \\
&= \sum_{s=1}^{r}\sum_{t=1}^{r} \alpha_{ks}\alpha_{lt} \left\langle f_s f_t \cdot \phi, \phi \right\rangle \\
&= 0.
\end{aligned}$$

For $r = 4$, the volume form is invariant under $SO(4)$, $f_1' f_2' f_3' f_4' = f_1 f_2 f_3 f_4$, and

$$\left\langle f_1' f_2' f_3' f_4' \cdot \phi, \phi \right\rangle = \left\langle f_1 f_2 f_3 f_4 \cdot \phi, \phi \right\rangle = 0.$$

□

Lemma 3.3 *Given a partially pure spinor $\phi \in \Sigma_r \otimes \Delta_n$, there exists an orthogonal complex structure on V^ϕ and $n - r \equiv 0 \pmod 2$.*

Proof. By definition, for every $X \in V^\phi$, there exists $Y \in V^\phi$ such that

$$X \cdot \phi = iY \cdot \phi,$$

and

$$Y \cdot \phi = i(-X) \cdot \phi.$$

If we set

$$J^\phi(X) := Y,$$

we get a linear transformation $J^\phi : V^\phi \to V^\phi$, such that $(J^\phi)^2 = -\mathrm{Id}_{V^\phi}$, i.e. J^ϕ is a complex structure on the vector space V^ϕ and $\dim_{\mathbb{R}}(V^\phi)$ is even. Furthermore, this complex structure is orthogonal. Indeed, for every $X \in V^\phi$,

$$\begin{aligned} X \cdot JX \cdot \phi &= -i|X|^2\phi, \\ JX \cdot X \cdot \phi &= i|JX|^2\phi, \end{aligned}$$

and

$$(-2\langle X, JX\rangle + i(|JX|^2 - |X|^2))\,\phi = 0,$$

i.e.

$$\begin{aligned} \langle X, JX\rangle &= 0 \\ |X| &= |JX|. \end{aligned}$$

$\square$

Lemma 3.4 *Let $r \geq 2$ and $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. The forms η^ϕ_{kl} are non-zero, $1 \leq k < l \leq r$.*

Proof. Since $(f_k f_l)^2 = -1$, the equation

$$\eta^\phi_{kl} \cdot \phi = -f_k f_l \cdot \phi \qquad (5)\ \boxed{\texttt{eq:despeje1}}$$

implies

$$\eta^\phi_{kl} \cdot f_k f_l \cdot \phi = \phi. \qquad (6)\ \boxed{\texttt{eq:despeje2}}$$

By taking an orthonormal frame $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ we can write

$$\eta^\phi_{kl} = \sum_{1\leq i<j\leq n} \eta^\phi_{kl}(e_i, e_j)e_i e_j.$$

By (6), and taking hermitian product with ϕ

$$1 = |\phi|^2$$

$$
\begin{aligned}
&= \left\langle \eta^{\phi}_{kl} \cdot f_k f_l \cdot \phi, \phi \right\rangle \\
&= \left\langle \sum_{1\leq i<j\leq n} \eta^{\phi}_{kl}(e_i,e_j) e_i e_j \cdot f_k f_l \cdot \phi, \phi \right\rangle \\
&= \sum_{1\leq i<j\leq n} \eta^{\phi}_{kl}(e_i,e_j) \langle e_i e_j \cdot f_k f_l \cdot \phi, \phi \rangle \\
&= \sum_{1\leq i<j\leq n} \eta^{\phi}_{kl}(e_i,e_j)^2 .
\end{aligned}
$$

□

⟨lemma:lie-algebra⟩ **Lemma 3.5** *Let $r \geq 2$. The image of the map associated to a partially pure spinor $\phi \in \Sigma_r \otimes \Delta_n$,*

$$
\begin{array}{ccc}
\bigwedge^2 \mathbb{R}^r & \longrightarrow & \mathrm{End}(\mathbb{R}^n) \\
f_{kl} & \mapsto & \hat{\eta}^{\phi}_{kl},
\end{array}
$$

forms a Lie algebra of endomorphisms isomorphic to $\mathfrak{so}(r)$.

Proof. Let $(e_1, \ldots, e_n)$ be an orthonormal frame of $\mathbb{R}^n$. First, let us consider the following calculation for $i \neq j$, $k \neq l$, $s \neq t$:

$$
\begin{aligned}
\mathrm{Re}\left\langle e_s e_t \cdot \eta^{\phi}_{ij} \cdot f_k f_l \cdot \phi, \phi \right\rangle &= \mathrm{Re}\left\langle e_s e_t \cdot \left(\sum_{a<b} \eta^{\phi}_{ij}(e_a,e_b) e_a e_b \right) \cdot f_k f_l \cdot \phi, \phi \right\rangle \\
&= \mathrm{Re} \sum_{a<b} \eta^{\phi}_{ij}(e_a,e_b) \langle e_s \cdot e_t \cdot e_a \cdot e_b \cdot f_k f_l \cdot \phi, \phi \rangle \\
&= -\sum_b \eta^{\phi}_{ij}(e_s,e_b)\eta^{\phi}_{kl}(e_b,e_t) + \sum_b \eta^{\phi}_{kl}(e_s,e_b)\eta^{\phi}_{ij}(e_b,e_t) \\
&= -\sum_b [\hat{\eta}^{\phi}_{kl}]_{tb}[\hat{\eta}^{\phi}_{ij}]_{bs} + \sum_b [\hat{\eta}^{\phi}_{ij}]_{tb}[\hat{\eta}^{\phi}_{kl}]_{bs} \\
&= [\hat{\eta}^{\phi}_{ij}, \hat{\eta}^{\phi}_{kl}]_{ts}
\end{aligned}
$$

is the entry in row t and column s of the matrix $[\hat{\eta}^{\phi}_{ij}, \hat{\eta}^{\phi}_{kl}]$.

Secondly, we prove that the endomorphisms $\hat{\eta}^{\phi}_{kl}$ satisfy the commutation relations of $\mathfrak{so}(r)$:

1. If $1 \leq i,j,k,l \leq r$ are all different,
$$[\hat{\eta}^{\phi}_{kl}, \hat{\eta}^{\phi}_{ij}] = 0. \tag{7}$$
eq:[kl,ij]=0

2. If $1 \leq i,j,k \leq r$ are all different,
$$[\hat{\eta}^{\phi}_{ij}, \hat{\eta}^{\phi}_{jk}] = -\hat{\eta}^{\phi}_{ik}. \tag{8}$$
[ij,jk]=-ik

To prove (7), note that by (5),

$$\eta^{\phi}_{ij} \cdot f_k f_l \cdot \phi = \eta^{\phi}_{kl} \cdot f_i f_j \cdot \phi, \tag{9}$$
eq:identity1

by (7)

$$\begin{aligned}\mathrm{Re}\left\langle e_s e_t \cdot \eta_{ij}^{\phi} \cdot f_k f_l \cdot \phi, \phi \right\rangle &= [\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{kl}^{\phi}]_{ts}, \\ \mathrm{Re}\left\langle e_s e_t \cdot \eta_{kl}^{\phi} \cdot f_i f_j \cdot \phi, \phi \right\rangle &= [\hat{\eta}_{kl}^{\phi}, \hat{\eta}_{ij}^{\phi}]_{ts},\end{aligned}$$

and by (9) and the anticommutativity of the bracket,

$$[\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{kl}^{\phi}] = 0.$$

To prove (8), note that by (5)

$$f_i f_j \cdot \eta_{jk}^{\phi} \cdot \phi = f_i f_k \cdot \phi$$

and

$$f_j f_k \cdot \eta_{ij}^{\phi} \cdot \phi = -f_i f_k \cdot \phi$$

so that

$$f_j f_k \cdot \eta_{ij}^{\phi} \cdot \phi = f_i f_j \cdot \eta_{jk}^{\phi} \cdot \phi - 2 f_i f_k \cdot \phi.$$

Thus,

$$\mathrm{Re}\left\langle e_s e_t \cdot \eta_{ij}^{\phi} \cdot f_j f_k \cdot \phi, \phi \right\rangle = \mathrm{Re}\left\langle e_s e_t \cdot \eta_{jk}^{\phi} \cdot f_i f_j \cdot \phi, \phi \right\rangle - 2\eta_{ik}^{\phi}(e_s, e_t)$$

and by (7)

$$[\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{jk}^{\phi}] = [\hat{\eta}_{jk}^{\phi}, \hat{\eta}_{ij}^{\phi}] - 2\hat{\eta}_{ik}^{\phi},$$

i.e.

$$[\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{jk}^{\phi}] = -\hat{\eta}_{ik}^{\phi}.$$

Thirdly, we will prove, in five separate cases, that the set of endomorphisms $\{\hat{\eta}_{kl}^{\phi}\}$ is linearly independent. For $r = 0, 1$ there are no endomorphisms. For $r = 2$ it is obvious since there is only one non-zero endomorphism. For $r = 3$, suppose

$$0 = \alpha_{12}\hat{\eta}_{12}^{\phi} + \alpha_{13}\hat{\eta}_{13}^{\phi} + \alpha_{23}\hat{\eta}_{23}^{\phi},$$

where $\alpha_{12} \neq 0$. Take the Lie bracket with $\hat{\eta}_{13}^{\phi}$ to get

$$0 = \alpha_{12}\hat{\eta}_{23}^{\phi} - \alpha_{23}\hat{\eta}_{12}^{\phi},$$

i.e.

$$\hat{\eta}_{23}^{\phi} = \frac{\alpha_{23}}{\alpha_{12}}\hat{\eta}_{12}^{\phi}.$$

We can also consider the bracket with $\hat{\eta}^{\phi}_{23}$,

$$0 = -\alpha_{12}\hat{\eta}^{\phi}_{13} + \alpha_{13}\hat{\eta}^{\phi}_{12},$$

so that

$$\hat{\eta}^{\phi}_{13} = \frac{\alpha_{13}}{\alpha_{12}}\hat{\eta}^{\phi}_{12}.$$

By substituting in the original equation we get

$$0 = (\alpha_{12}^2 + \alpha_{13}^2 + \alpha_{23}^2)\hat{\eta}^{\phi}_{12},$$

which gives a contradiction.

Now suppose $r \geq 5$ and that there is a linear combination

$$0 = \sum_{k<l} \alpha_{kl}\hat{\eta}^{\phi}_{kl}.$$

Taking succesive brackets with $\hat{\eta}^{\phi}_{13}$, $\hat{\eta}^{\phi}_{12}$, $\hat{\eta}^{\phi}_{34}$ and $\hat{\eta}^{\phi}_{45}$ we get the identity

$$\alpha_{12}\hat{\eta}^{\phi}_{15} = 0,$$

i.e. $\alpha_{12} = 0$. Similar arguments give the vanishing of every α_{kl}.

For $r = 4$, suppose there is a linear combination

$$0 = \alpha_{12}\eta^{\phi}_{12} + \alpha_{13}\eta^{\phi}_{13} + \alpha_{14}\eta^{\phi}_{14} + \alpha_{23}\eta^{\phi}_{23} + \alpha_{24}\eta^{\phi}_{24} + \alpha_{34}\eta^{\phi}_{34}.$$

Multiply by $-\phi$

$$0 = (\alpha_{12}f_{12} + \alpha_{13}f_{13} + \alpha_{14}f_{14} + \alpha_{23}f_{23} + \alpha_{24}f_{24} + \alpha_{34}f_{34}) \cdot \phi.$$

Multiply by $-f_{12}$

$$0 = (\alpha_{12} - \alpha_{13}f_{23} - \alpha_{14}f_{24} + \alpha_{23}f_{13} + \alpha_{24}f_{14} - \alpha_{34}f_{1234}) \cdot \phi.$$

Now, take hermitian product with ϕ

$$\begin{aligned} 0 &= \langle(\alpha_{12} - \alpha_{13}f_{23} - \alpha_{14}f_{24} + \alpha_{23}f_{13} + \alpha_{24}f_{14} - \alpha_{34}f_{1234}) \cdot \phi, \phi\rangle \\ &= \alpha_{12}|\phi|^2 - \alpha_{34}\langle f_{1234} \cdot \phi, \phi\rangle \\ &= \alpha_{12}. \end{aligned}$$

Similar arguments give the vanishing of the other coefficients. □

⟨lemma:kernel⟩ **Lemma 3.6** *Let $r \geq 2$ and $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. Then*

$$V^{\phi} \subseteq \bigcap_{1\leq k<l\leq r} \ker \hat{\eta}^{\phi}_{kl}.$$

Proof. Let $1 \leq k < l \leq r$ be fixed and $X \in V^\phi$. Since $\mathbb{R}^n = V^\phi \oplus (V^\phi)^\perp$ and J^ϕ is a complex structure on V^ϕ, there exists a basis $\{e_1, e_2, \ldots, e_{2m-1}, e_{2m}\} \cup \{e_{2m+1}, \ldots, e_{2m+r}\}$ such that

$$\begin{aligned} V^\phi &= \mathrm{span}(e_1, e_2, \ldots, e_{2m-1}, e_{2m}), \\ (V^\phi)^\perp &= \mathrm{span}(e_{2m+1}, \ldots, e_{2m+r}), \\ J^\phi(e_{2j-1}) &= e_{2j}, \\ J^\phi(e_{2j}) &= -e_{2j-1}, \end{aligned}$$

where $m = (n-r)/2$ and $1 \leq j \leq m$. Note that

$$\begin{aligned} \hat{\eta}^\phi_{kl}(e_{2j-1}) &= \sum_{a=1}^n \mathrm{Re}\,\langle e_{2j-1} \wedge e_a \cdot f_{kl} \cdot \phi, \phi\rangle\, e_a \\ &= -\sum_{a \neq 2j-1}^n \mathrm{Re}\,\langle f_{kl} \cdot e_a e_{2j-1} \cdot \phi, \phi\rangle\, e_a \\ &= -\sum_{a \neq 2j-1}^n \mathrm{Re}\left\langle f_{kl} \cdot e_a (iJ^\phi(e_{2j-1})) \cdot \phi, \phi\right\rangle e_a \\ &= \sum_{a \neq 2j-1}^n \mathrm{Im}\,\langle e_a e_{2j} \cdot f_{kl} \cdot \phi, \phi\rangle\, e_a \\ &= -\mathrm{Im}\,\langle f_{kl} \cdot \phi, \phi\rangle\, e_{2j} \\ &= 0. \end{aligned}$$

□

Lemma 3.7 *Let $r \geq 2$ and $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. Then $(V^\phi)^\perp$ carries a standard representation of $\mathfrak{so}(r)$, and an orientation.*

Proof. By Lemma 3.5, $\mathfrak{so}(r)$ is represented non-trivially on $\mathbb{R}^n = V^\phi \oplus (V^\phi)^\perp$ and, by Lemma 3.6, it acts trivially on V^ϕ. Thus $(V^\phi)^\perp$ is a nontrivial representation of $\mathfrak{so}(r)$ of dimension r. □

Remark. The existence of a partially pure spinor implies $r \equiv n \pmod 2$. In this case, let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_r)$ be orthonormal frames for $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively,

$$\mathrm{vol}_n = e_1 \cdots e_n, \qquad \mathrm{vol}_r = f_1 \cdots f_r,$$

and

$$\begin{aligned} F : \Sigma_r \otimes \Delta_n &\longrightarrow \Sigma_r \otimes \Delta_n \\ \phi &\mapsto (-i)^{n/2} i^{r/2} \mathrm{vol}_n \cdot \mathrm{vol}_r \cdot \phi. \end{aligned}$$

Note that $i^{r/2}\mathrm{vol}_r$ acts as $(-1)^{r/2}\mathrm{Id}_{\Sigma_r}$ on Σ_r and that $(-i)^{n/2}\mathrm{vol}_n$ determines the decomposition $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$. Thus we have that

$$\Sigma_r \otimes \Delta_n = (\Sigma_r \otimes \Delta_n)^+ \oplus (\Sigma_r \otimes \Delta_n)^-,$$

and we will call elements in $(\Sigma_r \otimes \Delta_n)^+$ and $(\Sigma_r \otimes \Delta_n)^-$ positive and negative twisted spinors respectively.

Definition 3.2 *Let n be even, $\mathbb{R}^n$ be endowed with the standard inner product and orientation, and vol_n denote the volume form. Let V, W be two orthogonal oriented subspaces such that $\mathbb{R}^n = V \oplus W$. Furthermore, assume V admits a complex structure inducing the given orientation on V. The oriented triple (V, J, W) will be called* positive *if given (oriented) orthonormal frames $(v_1, J(v_1), \ldots, v_m, J(v_m))$ and $(w_1, \ldots, w_r)$ of V and W respectively,*

$$v_1 \wedge J(v_1) \wedge \ldots \wedge v_m \wedge J(v_m) \wedge w_1 \wedge \ldots \wedge w_r = \mathrm{vol}_n,$$

and negative *if*

$$v_1 \wedge J(v_1) \wedge \ldots \wedge v_m \wedge J(v_m) \wedge w_1 \wedge \ldots \wedge w_r = -\mathrm{vol}_n.$$

Lemma 3.8 *If r is even, a partially pure spinor ϕ is either positive or negative. Furthermore, a partialy pure spinor ϕ is positive (resp. negative) if and only if the corresponding oriented triple $(V^\phi, J^\phi, (V^\phi)^\perp)$ is positive (resp. negative).*

Proof. We must prove that either $\phi \in (\Sigma_r \otimes \Delta_n)^+$ or $\phi \in (\Sigma_r \otimes \Delta_n)^-$. Since ϕ is a partially pure spinor, there exist frames $(e'_1, \ldots, e'_{2m})$ and $(e'_{2m+1}, \ldots, e'_{2m+r})$ of V^ϕ and $(V^\phi)^\perp$ respectively such that

$$e'_{2j} = J(e'_{2j-1}) \quad \text{and} \quad \eta^\phi_{kl} = e'_{2m+k} \wedge e'_{2m+l},$$

where $1 \le j \le m$ and $1 \le k < l \le r$. Now,

$$e'_1 \wedge e'_2 \wedge \ldots \wedge e'_{2m} \wedge e'_{2m+1} \wedge \ldots \wedge e'_{2m+r} = \pm\mathrm{vol}_n.$$

Then,

$$\begin{aligned}
(-i)^{n/2} i^{r/2} \mathrm{vol}_n \cdot \mathrm{vol}_r \cdot \phi &= \pm(-i)^{n/2} i^{r/2} e'_1 e'_2 \cdots e'_{2m} e'_{2m+1} \cdots e'_{2m+r} \cdot f_1 \cdots f_r \cdot \phi \\
&= \pm(-i)^{n/2} i^{r/2} e'_1 J(e'_1) \cdots e'_{2m-1} J(e'_{2m-1}) \eta^\phi_{12} \cdots \eta^\phi_{r-3,r-2} \cdot f_{12} \cdots f_{r-1,r} \cdot \eta^\phi_{r-1,r} \cdot \phi \\
&= \pm(-i)^{n/2} i^{r/2} e'_1 J(e'_1) \cdots e'_{2m-1} J(e'_{2m-1}) \eta^\phi_{12} \cdots \eta^\phi_{r-3,r-2} \cdot f_{12} \cdots f_{r-3,r-2} \cdot \phi \\
&= \pm(-i)^{n/2} i^{r/2} e'_1 J(e'_1) \cdots e'_{2m-1} J(e'_{2m-1}) \cdot \phi \\
&= \pm(-i)^{n/2} i^{r/2} e'_1 J(e'_1) \cdots e'_{2m-3} J(e'_{2m-3}) e'_{2m-1} (-i e'_{2m-1}) \cdot \phi \\
&= \pm(-1)^m (-i)^{n/2+m} i^{r/2} \phi \\
&= \pm\phi,
\end{aligned}$$

i.e. $\phi \in (\Sigma_r \otimes \Delta_n)^\pm$. □

3.3 Orbit of a partially pure spinor

ally-pure-spinor⟩ **Lemma 3.9** *Let $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. If $g \in Spin^{c,r}(n)$, then $g(\phi)$ is also a partially pure spinor.*

Proof. Let $g \in Spin^{c,r}(n)$ and $\lambda^{c,r}_n(g) = (g_1, g_2, g_3) \in SO(n) \times SO(r) \times U(1)$. First, suppose $X, Y \in V^\phi$,

$$X \cdot \phi = i\, Y \cdot \phi.$$

Apply g on both sides

$$g_1(X) \cdot g(\phi) \quad = \quad i\, g_1(Y) \cdot g(\phi).$$

which means that g_1 maps V^ϕ into $V^{g(\phi)}$ injectively. On the other hand, any pair of vectors $\tilde{X}, \tilde{Y} \in V^{g(\phi)}$ such that

$$\tilde{X} \cdot g(\phi) = i\, \tilde{Y} \cdot g(\phi),$$

are the image under g_1 of some vectors $X, Y \in \mathbb{R}^n$, i.e.

$$g_1(X) \cdot g(\phi) = i\, g_1(Y) \cdot g(\phi).$$

Apply g^{-1} on both sides to get

$$X \cdot \phi = i\, Y \cdot \phi,$$

so that $X, Y \in V^\phi$, i.e. $V^{g(\phi)} = g_1(V^\phi)$. Moreover,

$$J^{g(\phi)} = g_1|_{V^\phi} \circ J^\phi \circ (g_1|_{v^\phi})^{-1}.$$

Now, let $e'_a = g_1^{-1}(e_a)$ and $f'_k = g_2^{-1}(f_k)$, so that

$$\begin{aligned}
\eta_{kl}^{g(\phi)} \cdot g(\phi) &= \sum_{1 \le a < b \le n} \eta_{kl}^{g(\phi)}(e_a, e_b) e_a e_b \cdot g(\phi) \\
&= \sum_{1 \le a < b \le n} \left\langle g_1(e'_a) g_1(e'_b) \cdot g_2(f'_k) g_2(f'_l) \cdot g(\phi), g(\phi) \right\rangle g_1(e'_a) g_1(e'_b) \cdot g(\phi) \\
&= \sum_{1 \le a < b \le n} \left\langle e'_a e'_b \cdot f'_k f'_l \cdot \phi, \phi \right\rangle g(e'_a e'_b \cdot \phi) \\
&= g\left(\sum_{1 \le a < b \le n} \eta'^{\phi}_{kl}(e'_a, e'_b) e'_a e'_b \cdot \phi \right) \\
&= g\left(\eta'^{\phi}_{kl} \cdot \phi \right) \\
&= g\left(-f'_k f'_l \cdot \phi \right) \\
&= -f_k f_l \cdot g(\phi),
\end{aligned}$$

and

$$\begin{aligned}
\langle f_k f_l \cdot g(\phi), g(\phi) \rangle &= \left\langle g(f'_k f'_l \cdot \phi), g(\phi) \right\rangle \\
&= \left\langle f'_k f'_l \cdot \phi, \phi \right\rangle \\
&= 0.
\end{aligned}$$

For $r = 4$, note that the volume form is invariant under $SO(4)$

$$\langle f_1 f_2 f_3 f_4 \cdot g(\phi), g(\phi) \rangle = \langle f_1 f_2 f_3 f_4 \cdot \phi, \phi \rangle = 0.$$

$\square$

Lemma 3.10 *Let $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. The stabilizer of ϕ is isomorphic to $U(m) \times SO(r)$.*

Proof. Let $g \in Spin^{c,r}(n)$ be such that $g(\phi) = \phi$ and $\lambda_n^{c,r}(g) = (g_1, g_2, g_3) \in SO(n) \times SO(r) \times U(1)$. It can be checked easily that

$$\begin{aligned} [g_1, J^\phi] &= 0 \\ g_1(V^\phi) &= V^\phi \\ g_1|_{V^\phi} &\in U(V^\phi, J^\phi) \cong U(m). \end{aligned}$$

Clearly, $g_1((V^\phi)^\perp) = (V^\phi)^\perp$.

As in Lemma 3.6, one can prove

$$\eta_{kl}^\phi = \sum_{2m+1 \le a < b \le 2m+r} \eta_{kl}^\phi(e_a, e_a) e_a e_b \in \bigwedge^2 (V^\phi)^\perp,$$

where $(e_1, \ldots, e_{2m+r})$ is an oriented frame of $V^\phi \oplus (V^\phi)^\perp$. Furthermore,

$$g_1(\eta_{kl}^\phi) = \eta_{kl}'^\phi,$$

where $f_k' = g_2(f_k)$. Now, we have that

$$\begin{array}{ccc} f_k f_l & \xrightarrow{g_2} & f_k' f_l' \\ \downarrow & & \downarrow \\ \eta_{kl}^\phi & \xrightarrow{h_2} & \eta_{kl}'^\phi \end{array}$$

for the diagram

$$\begin{array}{ccccccc} \mathfrak{so}(r) & \cong & \bigwedge^2 \mathbb{R}^r & \xrightarrow{g_2} & \bigwedge^2 \mathbb{R}^r & \cong & \mathfrak{so}(r) \\ & & \downarrow & & \downarrow & & \\ \mathfrak{so}(r) & \cong & \bigwedge^2 (V^\phi)^\perp & \xrightarrow{h_2} & \bigwedge^2 (V^\phi)^\perp & \cong & \mathfrak{so}(r) \end{array}$$

where the vertical arrows are Lie algebra isomorphisms and the horizontal arrows correspond to g_2 and h_2 acting via the adjoint representation of $SO(r)$. Thus, h_2 and g_2 correspond to each other under the isomorphism $\bigwedge^2 (V^\phi)^\perp \cong \bigwedge^2 \mathbb{R}^r$ given by $f_{kl} \mapsto \eta_{kl}^\phi$.

Since h_1 is unitary with respect to J, there is a frame $(e_1, \ldots, e_{2m})$ of V^ϕ such that

$$e_{2j} = J(e_{2j-1})$$

and h_1 is diagonal with respect to the unitary basis $\{e_{2j-1} - ie_{2j} \,|\, j = 1, \ldots, m\}$, i.e.

$$h_1(e_{2j-1} - ie_{2j}) = e^{i\theta_j}(e_{2j-1} - ie_{2j})$$

where $0 \le \theta_j < 2\pi$. On the other hand, there is a frame $(f_1, \ldots, f_r)$ of $\mathbb{R}^r$ such that

$$g_2 = R_{\varphi_1} \circ \cdots \circ R_{\varphi_{[r/2]}}$$

where R_{φ_k} is a rotation by an angle φ_k on the plane generated by f_{2k-1} and f_{2k}, $1 \leq k \leq [r/2]$. Now, since the endomorphisms $\hat{\eta}^{\phi}_{kl}$ span an isomorphic copy of $\mathfrak{so}(r)$, there is a frame $(e_{2m+1}, \ldots, e_{2m+r})$ of $(V^{\phi})^{\perp}$ such that

$$\eta^{\phi}_{kl} = e_{2m+k} \wedge e_{2m+l},$$

$1 \leq k < l \leq r$. Since the adjoint representation of $SO(r)$ is faithful

$$h_2 = R'_{\varphi_1} \circ \cdots \circ R'_{\varphi_{[r/2]}}$$

where R'_{φ_k} is a rotation by an angle φ_k on the plane generated by $e_{2m+2k-1}$ and e_{2m+2k}, $1 \leq k \leq [r/2]$. Thus,

$$\begin{aligned} g \;\; = \;\; & \pm\Bigg[\prod_{j=1}^{m}(\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j}) \cdot \prod_{k=1}^{[r/2]}(\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^{\phi}_{2k-1,2k}), \\ & \prod_{k=1}^{[r/2]}(\cos(\varphi_k/2) - \sin(\varphi_k/2)f_{2k-1}f_{2k}), e^{i\theta/2}\Bigg]. \end{aligned}$$

Now,

$$\begin{aligned} \phi \;\; &= \;\; g(\phi) \\ &= \;\; \pm e^{i\theta/2}\prod_{j=1}^{m}(\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j}) \\ & \quad \cdot \prod_{k=1}^{[r/2]}(\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^{\phi}_{2k-1,2k}) \cdot \prod_{k=1}^{[r/2]}(\cos(\varphi_k/2) - \sin(\varphi_k/2)f_{2k-1}f_{2k})(\phi) \\ &= \;\; \pm e^{i\theta/2}\prod_{j=1}^{m}(\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j}) \cdot \prod_{k=1}^{[r/2]-1}(\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^{\phi}_{2k-1,2k}) \\ & \quad \cdot \prod_{k=1}^{[r/2]}(\cos(\varphi_k/2) - \sin(\varphi_k/2)f_{2k-1}f_{2k})(\cos(\varphi_{[r/2]}/2) - \sin(\varphi_{[r/2]}/2)\eta^{\phi}_{2[r/2]-1,2[r/2]}) \cdot (\phi) \\ &= \;\; \pm e^{i\theta/2}\prod_{j=1}^{m}(\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j}) \\ & \quad \cdot \prod_{k=1}^{[r/2]-1}(\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^{\phi}_{2k-1,2k}) \cdot \prod_{k=1}^{[r/2]-1}(\cos(\varphi_k/2) - \sin(\varphi_k/2)f_{2k-1}f_{2k}) \\ & \quad (\cos(\varphi_{[r/2]}/2) - \sin(\varphi_{[r/2]}/2)f_{2[r/2]-1}f_{2[r/2]}) \cdot (\cos(\varphi_{[r/2]}/2) + \sin(\varphi_{[r/2]}/2)f_{2[r/2]-1}f_{2[r/2]}) \cdot (\phi) \\ &= \;\; \pm e^{i\theta/2}\prod_{j=1}^{m}(\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j}) \\ & \quad \cdot \prod_{k=1}^{[r/2]-1}(\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^{\phi}_{2k-1,2k}) \cdot \prod_{k=1}^{[r/2]-1}(\cos(\varphi_k/2) - \sin(\varphi_k/2)f_{2k-1}f_{2k})(\phi) \end{aligned}$$

$$
\begin{aligned}
&= \pm e^{i\theta/2}\prod_{j=1}^{m}(\cos(\theta_j/2)-\sin(\theta_j/2)e_{2j-1}e_{2j})(\phi)\\
&= \pm e^{i\theta/2}\prod_{j=1}^{m}(\cos(\theta_j/2)+i\sin(\theta_j/2)e_{2j-1}(iJ(e_{2j-1})))(\phi)\\
&= \pm e^{i\theta/2}\prod_{j=1}^{m}(\cos(\theta_j/2)+i\sin(\theta_j/2)e_{2j-1}e_{2j-1})(\phi)\\
&= \pm e^{i\theta/2}\prod_{j=1}^{m}(\cos(\theta_j/2)-i\sin(\theta_j/2))(\phi)\\
&= \pm e^{i\theta/2}\prod_{j=1}^{m}e^{-i\theta_j/2}(\phi)\\
&= \pm e^{\frac{i}{2}(\theta-\sum_{j=1}^{m}\theta_j)}(\phi).
\end{aligned}
$$

This means

$$e^{\frac{i}{2}(\theta-\sum_{j=1}^{m}\theta_j)}=\pm 1$$

i.e.

$$
\begin{aligned}
\det_{\mathbb{C}}(h_1) &= e^{i\sum_{j=1}^{m}\theta_j}\\
&= e^{i\theta}.
\end{aligned}
$$

Thus we have found that

$$\lambda_n^{c,r}(g)=((h_1,h_2),h_2,\det_{\mathbb{C}}(h_1)),$$

which is in the image of the horizontal row in the diagram of Lemma 2.4

$$
\begin{tikzcd}
 & Spin^{c,r}(n) \arrow[d] \\
U(m)\times SO(r) \arrow[r] \arrow[ur] & SO(n)\times SO(r)\times U(1)
\end{tikzcd}
$$

□

Remark. Note that for any spinor $\phi\in\Sigma_r\otimes\Delta_n$, $g\in Spin^{c,r}(n)$, $\lambda_n^{c,r}(g)\in SO(n)\times SO(r)\times U(1)$,

$$
\begin{aligned}
\eta_{kl}^{g(\phi)}(X,Y) &= \langle X\wedge Y\cdot f_kf_l\cdot g(\phi),g(\phi)\rangle\\
&= \left\langle g_1(X')\wedge g_1(Y')\cdot g_2(f_k')g_2(f_l')\cdot g(\phi),g(\phi)\right\rangle\\
&= \left\langle g(X'\wedge Y'\cdot f_k'f_l'\cdot\phi),g(\phi)\right\rangle\\
&= \left\langle X'\wedge Y'\cdot f_k'f_l'\cdot\phi,\phi\right\rangle\\
&=: \eta_{kl}'^{\phi}(X',Y'),
\end{aligned}
$$

for $X'=g_1^{-1}(X),Y'=g_1^{-1}(Y)\in\mathbb{R}^n$, $f_k'=g_2^{-1}(f_k)$. Thus, the matrices representing $\eta_{kl}^{g(\phi)}$ (with respect to some basis) are conjuagte to the matrices representing $\eta_{kl}'^{\phi}$.

ma: little orbit⟩ **Lemma 3.11** *Let $\phi, \psi \in \Sigma_r \otimes \Delta_n$ be partially pure spinors and $Spin^c(r)$ the standard copy of this group in $Spin^{c,r}(n)$. Then, $\psi \in Spin^c(r) \cdot \phi$ if and only if they generate the same oriented tiple $(V^\phi, J^\phi, (V^\phi)^\perp) = (V^\psi, J^\psi, (V^\psi)^\perp)$.*

Proof. Suppose $\psi = g(\phi)$ for some $g \in Spin^c(r) \subset Spin^{c,r}(n)$, and let $\lambda_n^{c,r}(g) = (1, g_2, e^{i\theta})$. Such an element induces

$$\langle X \wedge Y \cdot f_k f_l \cdot g(\phi), g(\phi)\rangle \quad = \quad \left\langle X \wedge Y \cdot f_k' f_l' \cdot \phi, \phi \right\rangle$$

for $f_k' = g_2^{-1}(f_k)$, i.e.

$$\eta_{kl}^{g(\phi)}(X,Y) = \eta_{kl}'^{\phi}(X,Y),$$

so that they span the same copy of $\mathfrak{so}(r)$ in $\mathrm{End}^-(\mathbb{R}^n)$,

$$\mathrm{span}(\eta_{kl}^{g(\phi)}) = \mathrm{span}(\eta_{kl}'^{\phi}) \cong \mathfrak{so}(r) \subset \mathrm{End}^-(\mathbb{R}^n).$$

Thus, by Lemma 3.9, the partially pure spinors ϕ and $g(\phi)$ determine the same oriented triple $(V^{g(\phi)}, J^{g(\phi)}, (V^{g(\phi)})^\perp) = (V^\phi, V^\phi, (V^\phi)^\perp)$.

Conversely, assume $(V^\phi, J^\phi, (V^\phi)^\perp) = (V^\psi, J^\psi, (V^\psi)^\perp)$, and consider the subalgebras of

$$\begin{aligned}
\mathfrak{so}(r)^\phi &= \mathrm{span}(\eta_{kl}^\phi + f_{kl}) \\
\mathfrak{so}(r)^\psi &= \mathrm{span}(\eta_{kl}^\psi + f_{kl}).
\end{aligned}$$

There exist frames $(e_{2m+1}, \ldots, e_{2m+r})$ and $(e_{2m+1}', \ldots, e_{2m+r}')$ of $(V^\phi)^\perp$ and $(V^\psi)^\perp$ respectively, such that

$$\begin{aligned}
\eta_{kl}^\phi &= e_{2m+k} \wedge e_{2m+l}, \\
\eta_{kl}^\psi &= e_{2m+k}' \wedge e_{2m+l}'.
\end{aligned}$$

Let $A = (a_{kl}) \in SO(r)$ the matrix such that

$$A:\ e_{2m+k}' \mapsto a_{k1} e_{2m+1}' + \cdots + a_{kr} e_{2m+r}' = e_{2m+k}$$

$1 \leq k < l \leq r$. The induced transformation maps

$$A:\ e_{2m+k}' \wedge e_{2m+l}' \mapsto e_{2m+k} \wedge e_{2m+l},$$

and set

$$A^T:\ f_k \mapsto a_{1k} f_1 + \cdots + a_{rk} f_r = f_k',$$

and

$$A^T:\ f_k \wedge f_l \mapsto f_k' \wedge f_l'.$$

Consider

$$\left\langle e_{2m+p} \wedge e_{2m+q} \cdot f_k' f_l' \cdot \psi, \psi \right\rangle \quad = \quad \left\langle \left(\sum_{s=1}^r a_{ps} e_{2m+s}' \right) \wedge \left(\sum_{t=1}^r a_{qt} e_{2m+t}' \right) \cdot f_k' f_l' \cdot \psi, \psi \right\rangle$$

$$
\begin{aligned}
&= \left\langle \left(\sum_{s<t} (a_{ps}a_{qt} - a_{pt}a_{qs}) e'_{2m+s} \wedge e'_{2m+t} \right) \cdot f'_k f'_l \cdot \psi, \psi \right\rangle \\
&= \sum_{s<t} (a_{ps}a_{qt} - a_{pt}a_{qs}) \left\langle e'_{2m+s} \wedge e'_{2m+t} \cdot f'_k f'_l \cdot \psi, \psi \right\rangle \\
&= \sum_{s<t} (a_{ps}a_{qt} - a_{pt}a_{qs}) \left\langle e'_{2m+s} \wedge e'_{2m+t} \cdot \left(\sum_{i=1}^{r} a_{ik} f_i \right) \left(\sum_{j=1}^{r} a_{jl} f_j \right) \cdot \psi, \psi \right\rangle \\
&= \sum_{s<t} (a_{ps}a_{qt} - a_{pt}a_{qs}) \left\langle e'_{2m+s} \wedge e'_{2m+t} \cdot \left(\sum_{i<j} (a_{ik}a_{jl} - a_{il}a_{jk}) f_i f_j \right) \cdot \psi, \psi \right\rangle \\
&= \sum_{s<t} \sum_{i<j} (a_{ps}a_{qt} - a_{pt}a_{qs})(a_{ik}a_{jl} - a_{il}a_{jk}) \left\langle e'_{2m+s} \wedge e'_{2m+t} \cdot f_i f_j \cdot \psi, \psi \right\rangle \\
&= \sum_{s<t} \sum_{i<j} (a_{ps}a_{qt} - a_{pt}a_{qs})(a_{ik}a_{jl} - a_{il}a_{jk}) \delta_{si}\delta_{tj} \\
&= \sum_{s<t} (a_{ps}a_{qt} - a_{pt}a_{qs})(a_{sk}a_{tl} - a_{sl}a_{tk}) \\
&= \delta_{pk}\delta_{ql},
\end{aligned}
$$

since the induced tranformation by A on $\bigwedge^2 \mathbb{R}^r$ is orthogonal. This means

$$\eta'^{\psi}_{kl} = \eta^{\phi}_{kl} = e_{2m+k} \wedge e_{2m+l}.$$

Now consider $g \in Spin^c(r) \subset Spin^{c,r}(n)$ such that $\lambda_n^{c,r}(g) = (1, A, 1) \in SO(n) \times SO(r) \times U(1)$. Then

$$
\begin{aligned}
\delta_{pk}\delta_{ql} &= \left\langle e_{2m+p} \wedge e_{2m+q} \cdot f'_k f'_l \cdot \psi, \psi \right\rangle \\
&= \left\langle g(e_{2m+p} \wedge e_{2m+q} \cdot f'_k f'_l \cdot \psi), g(\psi) \right\rangle \\
&= \left\langle e_{2m+p} \wedge e_{2m+q} \cdot A(f'_k) A(f'_l) \cdot g(\psi), g(\psi) \right\rangle \\
&= \left\langle e_{2m+p} \wedge e_{2m+q} \cdot f_k f_l \cdot g(\psi), g(\psi) \right\rangle,
\end{aligned}
$$

i.e.

$$\eta^{g(\psi)}_{kl} = e_{2m+k} \wedge e_{2m+l} = \eta^{\phi}_{kl},$$

so that

$$\mathfrak{so}(r)^{g(\psi)} = \mathrm{span}(\eta^{g(\psi)}_{kl} + f_{kl}) = \mathrm{span}(\eta^{\phi}_{kl} + f_{kl}) = \mathfrak{so}(r)^{\phi}.$$

This implies that $g(\psi)$ and ϕ share the same stabilizer

$$U(V^{\phi}, J^{\phi}) \times exp(\mathfrak{so}(r)^{\phi}) = U(V^{g(\psi)}, J^{g(\psi)}) \times exp(\mathfrak{so}(r)^{g(\psi)}) \cong U(m) \times SO(r).$$

But there is only a 1-dimensional summand in the decomposition of $\Sigma_r \otimes \Delta_n$ under this subgroup. More precisely, under this subgroup

$$\Sigma_r \otimes \Delta_n = \Sigma_r \otimes \Delta_r \otimes \Delta_{2m},$$

where Δ_{2m} decomposes under $U(m)$ and contains only a 1-dimensional trivial summand [3], while $\Sigma_r \otimes \Delta_r$ is isomorphic to a subspace of the complexified space of alternating forms on $\mathbb{R}^r$ which also contains only a 1-dimensional trivial summand. Thus, $g(\psi) = e^{i\theta}\phi$ for some $\theta \in [0, 2\pi) \subset \mathbb{R}$. □

Lemma 3.12
- *If r is odd, the group $Spin^{c,r}(n)$ acts transtitively on the set of partially pure spinors in $\Sigma_r \otimes \Delta_n$.*
- *If r is even, the group $Spin^{c,r}(n)$ acts transtitively on the set of positive partially pure spinors in $(\Sigma_r \otimes \Delta_n)^+$.*

Proof. Suppose that r is odd. Note that the standard partially pure spinor ϕ_0 satisfies the conditions

$$\left\{\begin{array}{rcl} e_{2j-1}e_{2j}\cdot\phi_0 &=& i\phi_0, \\ e_{2m+k}e_{2m+l}\cdot\phi &=& -f_{kl}\cdot\phi, \\ \langle f_{kl}\cdot\phi,\phi\rangle &=& 0, \end{array}\right. \tag{10}$$

eq: equations s

where $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_r)$ are the standard oriented frames of $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. There exist frames $(e'_1, \ldots, e'_{2m})$ and $(e'_{2m+1}, \ldots, e'_{2m+r})$ of V^ϕ and $(V^\phi)^\perp$ respectively such that

$$e'_{2j} = J(e'_{2j-1}) \quad \text{and} \quad \eta^\phi_{kl} = e'_{2m+k} \wedge e'_{2m+l},$$

$1 \leq k < l \leq r$, $1 \leq j \leq m$. Call $g'_1 \in O(n)$ the transformation of $\mathbb{R}^n$ taking the new frame to the standard one. Define $g_1 \in SO(n)$ as follows

$$\left\{\begin{array}{ll} g_1 = g'_1, & \text{if } e'_1 \wedge \ldots \wedge e'_{2m+r} = \mathrm{vol}_n, \\ g_1 = -g'_1, & \text{if } e'_1 \wedge \ldots \wedge e'_{2m+r} = -\mathrm{vol}_n. \end{array}\right.$$

Then $(g_1, 1, 1) \in SO(n) \times SO(r) \times U(1)$ has two preimages $\pm\tilde{g} \in Spin^{c,r}(n)$. By Lemma 3.9, $\tilde{g}(\phi)$ is a partially pure spinor. We will check that $\tilde{g}(\phi)$ satisfies (10) as ϕ_0 does. Indeed,

$$\begin{array}{rcl} e_{2j-1}e_{2j}\cdot\tilde{g}(\phi) &=& g'_1(e'_{2j-1})g'_1(e'_{2j})\cdot\tilde{g}(\phi) \\ &=& (\pm g_1(e'_{2j-1}))(\pm g_1(e'_{2j}))\cdot\tilde{g}(\phi) \\ &=& g_1(e'_{2j-1})g_1(e'_{2j})\cdot\tilde{g}(\phi) \\ &=& \tilde{g}(e'_{2j-1}e'_{2j}\cdot\phi) \\ &=& \tilde{g}(i\phi) \\ &=& i\tilde{g}(\phi), \end{array}$$

and

$$\begin{array}{rcl} e_{2m+k}e_{2m+l}\cdot\tilde{g}(\phi) &=& g'_1(e'_{2m+k})g'_1(e'_{2m+l})\cdot\tilde{g}(\phi) \\ &=& (\pm g_1(e'_{2m+k}))(\pm g_1(e'_{2m+l}))\cdot\tilde{g}(\phi) \\ &=& g_1(e'_{2m+k})g_1(e'_{2m+l})\cdot\tilde{g}(\phi) \\ &=& \tilde{g}(e'_{2m+k}e'_{2m+l}\cdot\phi) \\ &=& \tilde{g}(-f_kf_l\cdot\phi) \end{array}$$

$$
\begin{aligned}
&= -\lambda_2(\tilde{g})(f_k)\lambda_2(\tilde{g})(f_l)\cdot\tilde{g}(\phi) \\
&= -f_kf_l\cdot\tilde{g}(\phi),
\end{aligned}
$$

since $\lambda_2(\tilde{g}) = 1$. Similarly,

$$
\begin{aligned}
\langle f_kf_l\cdot\tilde{g}(\phi),\tilde{g}(\phi)\rangle &= \langle\lambda_2(\tilde{g})(f_k)\lambda_2(\tilde{g})(f_l)\cdot\tilde{g}(\phi),\tilde{g}(\phi)\rangle \\
&= \langle\tilde{g}(f_kf_l\cdot\phi),\tilde{g}(\phi)\rangle \\
&= \langle f_kf_l\cdot\phi,\phi\rangle \\
&= 0.
\end{aligned}
$$

Thus, $\tilde{g}(\phi)$ generates the same oriented triple $(V^{\tilde{g}(\phi)}, J^{\tilde{g}(\phi)}, (V^{\tilde{g}(\phi)})^\perp) = (V^{\phi_0}, J^{\phi_0}, (V^{\phi_0})^\perp)$ as ϕ_0 which, by Lemma 3.11, concludes the proof for r odd.

The case for r even is similar. ☐

characterization⟩ **Theorem 3.1** *Let $\mathbb{R}^n$ be endowed with the standard inner product and orientation. Given $r\in\mathbb{N}$ such that $r<n$, the following objects are equivalent:*

1. *A (positive) triple consisting of a codimension r vector subspace endowed with an orthogonal complex structure and an oriented orthogonal complement.*
2. *An orbit $Spin^c(r)\cdot\phi$ for some (positive) twisted partially pure spinor $\phi\in\Delta_n\otimes\Sigma_r$.*

Proof. Given a codimension r vector subspace D endowed with an orthogonal complex structure, $\dim_{\mathbb{R}}(D) = 2m$, $n = 2m + r$. By Lemma 2.5

$$\Delta_n \cong \Delta(D^\perp)\otimes\Delta(D).$$

Let us define

$$
\Sigma_r \quad\cong\quad \begin{cases} \Delta(D^\perp) & \text{if } r \text{ is odd,} \\ \Delta(D^\perp)^+ & \text{if } r \text{ is even,} \end{cases}
$$

so that

$$\Sigma_r\otimes\Delta_n$$

contains the standard twisted partially pure spinor ϕ_0 of Lemma 3.1.

The proof of the converse is the content of Subsection 3.2. ☐

Let $\tilde{\mathcal{S}}$ denote the set of all partially pure spinors of rank r

$$\tilde{\mathcal{S}} = \frac{Spin^{c,r}(n)}{U(m)\times SO(r)}.$$

Consider

$$\mathcal{S} = \frac{\tilde{\mathcal{S}}}{Spin^c(r)}$$

where $Spin^c(r)$ is the canonical copy of such a group in $Spin^{c,r}(n)$. Thus we have the following expected result.

Corollary 3.1 *The space parametrizing subspaces with orthogonal complex structures of codimension r in $\mathbb{R}^n$ with oriented orthogonal complements is*

$$\mathcal{S} \cong \frac{SO(n)}{U(m) \times SO(r)}.$$

□

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